ACCELERATOR ALIGNMENT
A PROBLEM IN FEEDBACK-CONTROL SYSTEMS*

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Summary

Obtaining and keeping a satisfactory closed beam orbit is a major problem in the design of large accelerators. The system determining this orbit has many variables, these variables are distributed around the accelerator structure, and the existence of the beam is periodic.

The designer may attempt to ensure satisfactory values for these variables by brute-force techniques such as ultra-precision construction, or he may apply the more subtle techniques of system control theory. In any case, he is dealing with a multivariable, interacting, sampled-data control problem. For the accelerators of the future, it may not be obvious that a solution exists.

In fact, however, there are techniques whereby an equilibrium orbit may first be obtained and thereafter maintained in the presence of all expected disturbances, by means of an electro-mechanical control system employing beam-position monitors. These techniques further permit a wide variety of operator experiments, such as trying various orbits, or restricting the spatial harmonic content of magnet re-positioning.

The purpose of this paper is to illustrate certain of these techniques. For clarity we will restrict our discussion to an elementary magnet-position control system for maintaining a closed orbit in the presence of magnet support settling. We assume the beam to be continously present, rather than to be discrete with time. This and other errors, such as magnet imperfections, and other control variables, such as backleg windings, would be handled in the same general way in a realistic design.

Our principle aim will be to show a technique whereby these interacting, multivariable control systems may be designed and the minimum expected performance bounded.

Explanation of Symbols

Capital letters and λ denote matrices.
Subscripts to capital letters denote the type of matrix.

A_n denotes the n x n matrix A_n.

A_d denotes the diagonal matrix A_d. This matrix is n x n, with all off-diagonal elements being zero.

A_c denotes the column matrix A_c. This matrix has one column, with n elements.

I_d denotes the identity matrix. This matrix has all elements on the diagonal of value unity, and all off-diagonal elements of value zero.

Lower case letters denote scalars unless subscripted. The letter s denotes the usual complex function; $s = \sigma + j\omega$

Lower case letters which are subscripted denote elements of matrices; a_{ij} denotes the element of matrix A_n lying in the i^{th} row and j^{th} column.

δ_{ik} denotes the Kronecker delta function:

$$\delta_{ik} = 1,\ i = k \qquad \delta_{ik} = 0,\ i \neq k$$

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A Magnet-Position Control Problem

Examine Figures 1 - 3. T_n is the matrix equation relating the displacement of the equilibrium orbit from the center of the vacuum chamber at n points to vertical movements of the n magnet support piles. As pointed out by Glen Lambertson and Jackson Laslett of U.C.R.L., T_n is in general nonsymmetric, and posseses zero and repeated eigenvalues. Therefore the inverse does not necessarily exist and its eigenvectors may not be linearly independent. However, by a similarity transformation, T_n may be brought into Jordan canonical form.

$$Q_n^{-1} T_n Q_n = \lambda_n'' = \lambda_d' + \lambda_n' \qquad (1)$$

λ_n' will be non-zero if T_n has some linearly dependent eigenvectors. Since (Figure 2)

$$A_n = T_n + B_n = (A_n^{-1})^{-1} \qquad (2)$$

the B_n in Figure 2 can be derived by

$$Q_n^{-1} A_n Q_n = \lambda_d = \lambda_n'' + \tilde{\lambda}_n \qquad (3)$$

$$Q_n^{-1} B_n Q_n = \tilde{\lambda}_n = \lambda_d - \lambda_n'' \qquad (3A)$$

This form of modification was first suggested by G. Lambertson since it allows strong conceptual reasoning when determining A_n.

Eigenvalue Form Aids Design

In particular, assume a disturbance P_c exists in Figure 2. The value of X_c which one would introduce would set $\Phi_c = 0$. Consider what this X_c must be and what the true beam error $\overset{\varepsilon}{\Phi}_c$ would be in consequence.

$$\Phi_c = 0 = T_n P_c + A_n X_c \tag{4}$$

$$X_c = -A_n^{-1} T_n P_c \tag{5}$$

$$\overset{\varepsilon}{\Phi}_c = T_n (P_c + X_c) \tag{6}$$

Using (1) - (3) in (5) and solving,

$$X_c = -Q_n (\lambda_d^{-1} \overset{''}{\lambda}_n) Q_n^{-1} P_c \tag{7}$$

Introducing (1), (7) and (3) into (6) and solving,

$$\overset{\varepsilon}{\Phi}_c = Q_n (\overset{''}{\lambda}_d^{-1} \overset{''}{\lambda}_n \overset{\sim}{\lambda}_n) Q_n^{-1} P_c \tag{8}$$

By setting

$$P_c = Q_n \underline{P}_c \tag{9}$$

We modify (7) and (8) to obtain

$$X_c = Q_n (\lambda_d^{-1} \overset{''}{\lambda}_n) \underline{P}_c \tag{10}$$

$$\overset{\varepsilon}{\Phi}_c = Q_n (\overset{''}{\lambda}_d^{-1} \overset{''}{\lambda}_n)(\overset{\sim}{\lambda}_n) \underline{P}_c \tag{11}$$

This enables the designer to solve for $\overset{\sim}{\lambda}$ in terms of the effect of the associated pseudo eigenvectors, Q_n.

Example. If T_n has no linearly dependent eigenvectors, as will probably be the case,

$\overset{''}{\lambda}_n = \overset{'}{\lambda}_d$, in (1)

$$\Phi_c = T_n P_c = Q_n \overset{'}{\lambda}_d Q_n^{-1} P_c = Q_n \overset{'}{\lambda}_d \underline{P}_c$$

$$X_c = -A_n^{-1} \Phi_c = -Q_n (\lambda_d^{-1} \overset{'}{\lambda}_d) \underline{P}_c, \text{ from (3)}$$

$$\underline{X}_c = Q_n X_c = -(\lambda_d^{-1} \overset{'}{\lambda}_d) \underline{P}_c$$

For very small $\overset{'}{\lambda}_{ii}$, $\underline{p}_i$ may be quite large and yet have little effect on the closed orbit. If $(\lambda_d^{-1} \overset{'}{\lambda}_d)$ were approximately I_d, $\underline{x}_i$ would have to make correspondingly large corrections, and yet accomplish little. Thus one might keep such corrections out of X_c by choosing

$\lambda_{ii} = 1$ when $|\overset{'}{\lambda}_{ii}| \ll 1$, using B_n of (3).

This is a reasonable approach, giving an inverse, and based on strong physical understanding.

The Control System Equations

In Figure 3, we show a control scheme intended to implement the desired solutions of (10) and (11). The final choice of $\overset{\sim}{\lambda}_n$ will depend upon the stability of this system, which will depend upon λ_d together with the accuracy with which the mathematical representation is physically approximated.

In designing the control system of Figure 3, we must include the effect of errors which may result in physically implementing the model.

Let us define:

Δ = Error matrix.

Also, let us substitute:

$$\text{for } T_n + B_n = A_n = Q_n \lambda_d Q_n^{-1}, \text{ use } \bar{A}_n \tag{12}$$

$$\text{where } \bar{A}_n = Q_n (\lambda_d + \overset{a}{\Delta}_n) Q_n^{-1} = Q_n \overset{\sim}{\lambda}_n Q_n^{-1} \tag{13}$$

$$\text{and for } (T_n + B_n)^{-1} = C_n \text{ use } \bar{C}_n \tag{14}$$

$$\text{where } \bar{C}_n = Q_n (\lambda_d^{-1} + \overset{c}{\Delta}_n) Q_n^{-1} \tag{15}$$

Including the effect of monitor and motor error, we define:

$$H_d = I_d + \overset{h}{I}_d = Q_n (I_d + \overset{h}{\Delta}_n) Q_n^{-1} \tag{16}$$

$$\frac{k}{s} K_d = \frac{k}{s} Q_n (I_d + \overset{k}{\Delta}_n) Q_n^{-1} \tag{17}$$

To simplify the expressions, let us further define:

$$R_c = 0 \tag{18}$$

$$K_d C_n H_d = M_n \tag{19}$$

$$M_n = Q_n (I_d + \overset{k}{\Delta}_n)(\lambda_d^{-1} + \overset{c}{\Delta}_n)(I_d + \overset{h}{\Delta}_n) Q_n^{-1} \tag{20}$$

Substituting (12) - (17) into Figure 3, and solving using (18) and (19), we find that

$$X_c(s) = -\left(\frac{s}{k} I_d + M_n \bar{A}_n\right)^{-1} M_n T_n P_c(s) \tag{21}$$

and

$$\overset{\varepsilon}{\Phi}_c(s) = T_n [P_c(s) + X_c(s)] \tag{6}$$

provided a stable inverse exists. Equation (21) is the system solution, and (6) the corresponding uncorrectable orbit error.

Example: Assume the system (21) is stable, and that

$$P_c(s) = 1/s\ P_c. \tag{22}$$

We can introduce (22) into (21) and apply the final value theorem,

$$\lim_{s \to 0} [s\ X_c(s)] = -\bar{A}_n^{-1}\ T_n P_c \tag{23A}$$

Using (13), (1) and (9), we find

$$\lim_{s \to 0} [sX_c(s)] = -Q_n(\lambda_d+\overset{a}{\Delta}_n)^{-1}\lambda''_n\ \underline{P}_c\ . \tag{23B}$$

We may now establish the allowable range of our errors $\overset{a}{\Delta}_n$ by the allowable range of X_c and $\overset{\varepsilon}{\Phi}_c$ in (10) and (11). It remains to show the constraints necessary for stability.

Eigenvalues and Stability

Inspecting (21), we see that for a stable inverse to exist, all poles of

$$(s/k\ I_d + M_n\ \bar{A}_n)^{-1} = F_n^{-1}(s) \tag{24}$$

must be in the left half of the s - plane. That is, if the p^{th} pole is defined as s_p, and if det is defined to mean determinant, all solutions s_p of the equation

$$\det(M_n\bar{A}_n + s/k\ I_d) = 0 \tag{25}$$

$$s = s_p;\ p = 1, \ldots, n$$

must be such that

$$\mathrm{Re}(s_p) = \mathrm{Re}(\sigma_p + j\omega_p) = \sigma_p < 0 \tag{26}$$

$$p = 1, \ldots, n.$$

If we multiply (20) and (13), and disregard all products of two or more error matrices as being of second order, we may substitute either

$$\Delta_n = \lambda_d^{-1}\overset{a}{\Delta}_n + \lambda_d^{-1}\ \overset{h}{\Delta}_n\ \lambda_d + \overset{c}{\Delta}_n\lambda_d + \overset{k}{\Delta}_n \tag{27}$$

or

$$E_n = A_n^{-1}\overset{a}{E}_n + A_n^{-1}\overset{h}{E}_d A_n + \overset{c}{E}_n A_n + \overset{k}{E}_d \tag{28}$$

in the product to obtain

$$M_n\bar{A}_n = Q_n[I_d + \Delta_n]Q_n^{-1} \tag{29}$$

$$\text{or } M_n\bar{A}_n = [I_d + E_n] \tag{30}$$

where the E denote the original error matrices before transformation.
Introducing (30) into (25) and simplifying, our stability condition is of the form

$$\det(E_n - \alpha I_d) = 0$$
$$\alpha = \alpha_p;\ p = 1, \ldots, n \tag{31}$$

where

$$s = -k(1 + \alpha) \tag{32}$$

$$\text{requires } \alpha_p > -1\ ;\ p = 1, \ldots, n \tag{33}$$

Equation (31) is seen to be a standard eigenvalue problem, and there exists a large library of programs for obtaining these eigenvalues, together with many approximation theorems for obtaining their bounds.

Example

In obtaining a complete solution, we might proceed as follows:

Obtain B_n

If we measure T_n in Figure 1, we must include the error $\overset{t}{E}_n$:

$$\bar{T}_n = T_n + \overset{t}{E}_n \tag{34}$$

The calculated A_n of (5) is then

$$A_n = B_n + \bar{T}_n \tag{35}$$

and this is used to derive C_n in (14). But the control solution will be implemented using the real T_n, so in (13) and (21),

$$\bar{A}_n = B_n + T_n\ . \tag{36}$$

If system (21) is stable, we may apply the final value theorem to obtain, for step disturbances $1/s\ P_c$,

$$\lim_{s \to 0}[s\ \overset{\varepsilon}{\Phi}_c(s)] = T_n[I_d - \bar{A}_n^{-1}T_n]P_c \tag{37A}$$

$$= T_n\ \bar{A}_n^{-1}\ B_n P_c \tag{37B}$$

Since we only know A_n, from (35), we must consider

$$T_n\bar{A}_n^{-1}\ B_n P_c = T_n A_n^{-1}[A_n\bar{A}_n^{-1}]B_n P_c \tag{38}$$

where

$$A_n\bar{A}_n^{-1} = A_n(A_n - \overset{t}{E}_n)^{-1} \tag{39A}$$

$$= (I_d - \overset{t}{E}_n\ A_n^{-1})^{-1} \tag{39B}$$

We then use

$$T_n A_n^{-1} B_n P_c \tag{40}$$

together with (3), (2), and (11), to design B_n, while placing constraints on (39B). That is, the inverse in (39B) must exist, and must have reasonable bounds.

Bound T_n Measurement Error

We assume the elements of $\overset{t}{E}_n$ in (34) to be mutually independent random variables with a common gaussian distribution, and the same standard deviation, $\overset{t}{\delta}$. We also calculate the standard deviation $\overset{1}{\delta}$ of the elements of A_n^{-1}. Consideration of (39B) might lead us to ask that

$$\sum_j^n \left| \sum_k^n \overset{t}{e}_{ik} \, a^{-1}_{kj} \right| < \ell; \; i = 1, \ldots, n \qquad (41)$$

with a probability of 0.95.

From the central limit theorem, this requires that

$$\sum_j^n \sum_k^n \overset{t}{e}_{ik} \, a^{-1}_{kj} = n(n^{\frac{1}{2}} \overset{t}{\delta} \overset{1}{\delta}) = \ell \qquad (42A)$$

and

$$\overset{t}{\delta} = n^{-3/2} \, \ell / \overset{1}{\delta} = \text{standard deviation of } \overset{t}{e}_{ij} \qquad (42B)$$

and

$$\overset{1}{\delta} = \text{standard deviation of elements in } A_n^{-1}. \qquad (42C)$$

One must consider the implication of (42) in deciding the final value for B_n, since from (35) it is clear that B_n will affect $\overset{1}{\delta}$ as well as a suitable ℓ. In general, one would like T_n to be measured quite accurately.

Bound the Stability

Among the many theorems which are available for bounding the eigenvalues of (31) is the Gersgorin Circle theorem. Considering (32) and (33) and the statistical nature of our errors, we may apply this theorem to (31) and obtain

$$|a| < \sum_j^n |e_{ij}| < 1 \; ; \; i = 1, \ldots, n \qquad (43)$$

In (28)

$$A_n^{-1} \overset{a}{E}_n = A_n^{-1} \overset{t}{E}_n = \overset{t}{E}_n \, A_n^{-1}. \qquad (44)$$

If we assume that all error matrices in (28) are defined statistically, as was $\overset{t}{E}_n$, with known standard deviations, we may again use the theory of random variables and the central limit theorem to obtain

$$\sum_j^n |e_{ij}| < n(n^{\frac{1}{2}} \overset{t}{\delta} \overset{1}{\delta} + \overset{1}{\delta} \overset{h}{\delta} \overset{2}{\delta} + n^{\frac{1}{2}} \overset{c}{\delta} \overset{2}{\delta}) + \overset{k}{\delta} < 1 \qquad (45)$$

where

$$\overset{2}{\delta} = \text{standard dev of elements in } A_n \qquad (46A)$$

$$\overset{h}{\delta} = \text{standard dev of elements in } \overset{h}{E}_d \qquad (46B)$$

$$\overset{k}{\delta} = \text{standard dev of elements in } \overset{k}{E}_d \qquad (46C)$$

When either n or the deviations are small, (45) can be used directly to establish stability. In general, the solution of (31) will require use of a computer, and the degree of difficulty will be about the same as determining a satisfactory B_n, as in (34) - (42).

Modify System Equations

We very briefly observe that (21) is by no means the only system equation which we might choose. For example, if we substitute

$$\left(\frac{s+h}{k}\right) \overset{k}{I}_d \text{ for } \frac{s}{k} \overset{k}{I}_d \qquad (47)$$

our stability conditions (33) and (45) become

$$a_p < -1 - h \qquad (48A)$$

$$\sum_j^n |e_{ij}| < (1+h) \qquad (49B)$$

which are clearly easier to satisfy for $h > 0$. Substituting (47) into (21) and applying the final value theorem, we see that the effect is to help stabilize by introducing less correction for a given error. This approach is common to all regulation systems in any discipline.

Determine k

Since k in (32) measures the speed with which the control system recovers from step errors, it is a measure of the power capacity of the motor drive system for changing the magnet support heights. Therefore, k should be no larger than necessary. For reasonable performance, the system should recover about three times as quickly as the fastest significant perturbation frequency of $P_c(t)$, f_p max.

$$\text{Thus } k = \frac{3(2\pi f_p \max)}{(1 - a \max)} \qquad (49)$$

is a reasonable choice.

Conclusion

This paper has attempted to give some indication of the techniques which one may use in obtaining a satisfactory equilibrium orbit by a control system which monitors the beam. I believe the general approach outlined here is practical, and that survey and foundation requirements are substantially reduced in consequence.

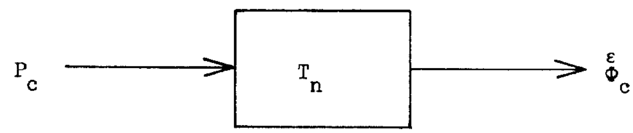

Fig. 1. Transfer function relating pile settling P_c to closed orbit displacement $\overset{\varepsilon}{\Phi}_c$ at n points around the ring. T_n may be non-symmetric, singular and possess small and repeated eigenvalues. T_n is also unalterable.

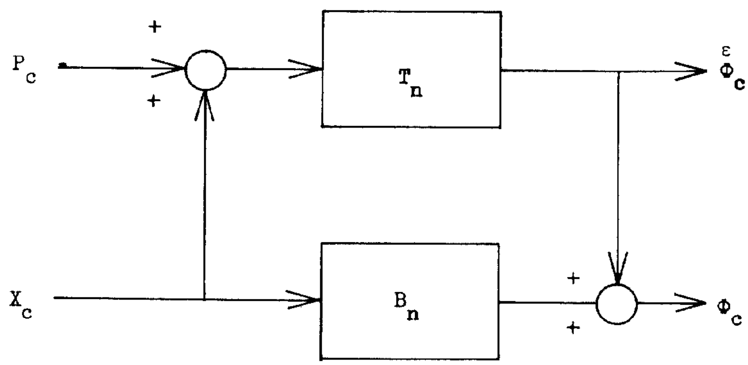

Fig. 2. Transfer function showing required modification of Fig. 1 so that pile cap corrective displacements X_c can be made. B_n is a corrective matrix introduced by the designer. B_n must be chosen so that $(T_n+B_n) = A_n$ where A_n has an inverse, and will generally represent a trade-off between stable regulation, physical range of X_c and minimum $\overset{\varepsilon}{\Phi}_c$.

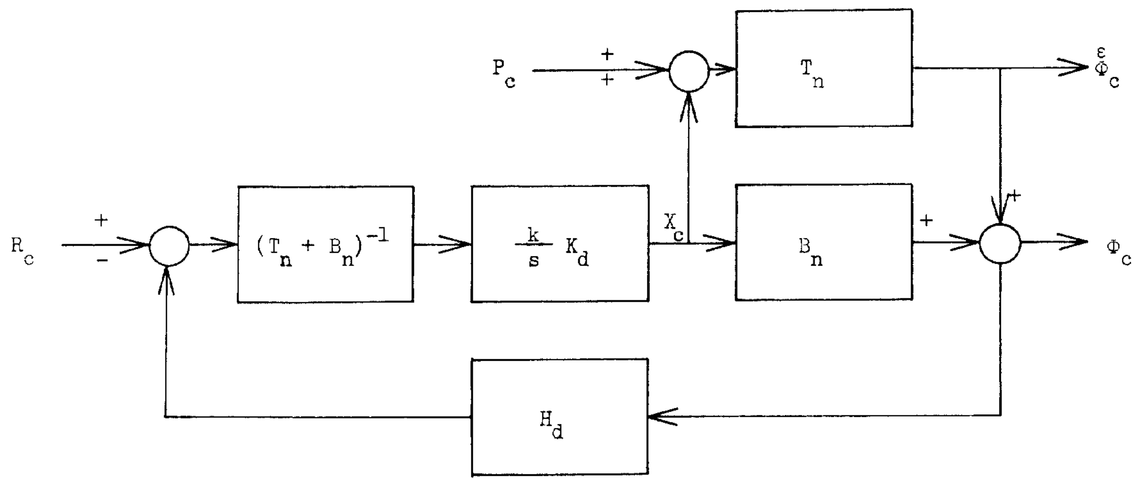

Fig. 3. Block diagram of the control system problem. While B_n is introduced to ensure that $\overset{\varepsilon}{\Phi}_c$ is uniquely and adequately related to X_c, Φ_c will only be bounded and these bounds will be weakened as $\overset{\varepsilon}{B}_n$ is strengthened. In general, the greater the allowable $\overset{\varepsilon}{\varphi}_i$ max, the more stable the resulting control system and the less range required in the magnet support adjustments X_c.